Assignment 12

This homework is due *Friday* Dec 6
(submissions as late as Tuesday Dec 10 will be accepted).

There are total 41 points in this assignment. 36 points is considered 100%. If you go over 36 points, you will get over 100% for this homework and it will count towards your course grade (up to 115%).

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge.* Your solutions should contain full proofs. Bare answers will not earn you much. This assignment covers sections 6.2, 6.4 in Bartle–Sherbert.

(1) [3pt] (6.2.3c) Find the points of relative extrema of the function $f(x) = x|x^2 - 12|$ for $-2 \le x \le 3$.

(2) [2pt] (6.2.6) Prove that $|\sin x - \sin y| \le |x - y|$ for all $x, y \in \mathbb{R}$. (Hint: Apply the Mean Value theorem to sin on the interval $[x, y]$.)

(3) (a) [3pt] (6.2.8) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Show that if $\lim_{x \to a} f'(x) = A$, then $f'(a)$ exists and is equal to A. (Hint: Use the limit definition of $f'(a)$ and apply the Mean Value Theorem to f on the interval $[a, x]$.)
(b) [3pt] Using Taylor decomposition, show that the function

$$f(x) = \begin{cases} \frac{1}{x} - \frac{1}{\sin x}, & x \in [-1, 1], x \neq 0; \\ 0, & x = 0, \end{cases}$$

is continuous at 0.
(c) [3pt] Prove that the function f above is differentiable at 0 and find the value of the derivative. (*Hint:* Use item 3a.)

(4) [2pt] (6.2.17) Let f, g be differentiable on $\mathbb{R}$ and suppose that $f(0) = g(0)$, and $f'(x) \le g'(x)$ for all $x \ge 0$. Show that $f(x) \le g(x)$ for all $x \ge 0$. (Hint: Apply the Mean Value Theorem to $f - g$ on $[0, x]$.)

(5) For a given function f and a point x_0, find Taylor's polynomials $P_2(x)$, $P_5(x)$, $P_{2013}(x)$ of $f(x)$ at x_0.
(a) [2pt] $f(x) = \sin x$ at $x_0 = \pi/2$. Compare to cos at 0.
(b) [2pt] $f(x) = \cos x$ at $x_0 = -\pi/2$. Compare to sin at 0.
(c) [2pt] $f(x) = x^3$ at $x_0 = 2$. Compare $P_3(x), P_5(x), P_{2013}(x)$ to $f(x)$.
(d) [2pt] $f(x) = \frac{1}{1-x}$ at $x_0 = 0$.
(e) [2pt] $f(x) = \frac{1}{x}$ at $x_0 = 1$. Compare to the previous item.
(You can take for granted that $(\sin x)' = \cos x, (\cos x)' = -\sin x$.)

(6) [3pt] (Part of exercise 6.4.7) If $x > 0$, show that

$$\left| \sqrt[3]{1+x} - \left(1 + \frac{1}{3}x - \frac{1}{9}x^2\right) \right| \le \frac{5}{81}x^3.$$

(*Hint:* Apply Taylor's Theorem to $f(x) = \sqrt[3]{1+x}$ with $n = 2$.)

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(7) (a) [2pt] Suppose $A \in \mathbb{R}$. Show that $\lim_{n\to\infty} \frac{A^n}{n!} = 0$.
Hint: take tail of this sequence that starts with $m > 2|A|$ and represent

$$\frac{A^n}{n!} = \frac{A^m}{m!} \cdot \frac{A^{n-m}}{(m+1)\cdots n}.$$

(b) [3pt] (6.4.8) If $f(x) = e^x$, show that the remainder term in Taylor's Theorem converges to zero as $n \to \infty$, for each fixed x_0 and x.

(c) [3pt] (6.4.9) If $g(x) = \cos x$, show that the remainder term in Taylor's Theorem converges to zero as $n \to \infty$, for each fixed x_0 and x.

(8) (6.4.14) Determine whether or not $x = 0$ is a point of relative extremum of the following functions:
(a) [2pt] $f(x) = x^3 + 2$,
(b) [2pt] $f(x) = \cos x - 1 + \frac{1}{2}x^2$.